

DYNAMIC ENTROPIC REPULSION FOR INTERACTING INTERFACES

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ABSTRACT. The dynamic entropic repulsion for the Ginzburg-Landau $\nabla\phi$ interface model was discussed in [4] and the asymptotics of the height of the interface was identified. This paper studies a similar problem for two interfaces on the wall which are interacting with one another by the exclusion rule. Each leading order of the asymptotics of height is $\sqrt{\log t}$ as $t \rightarrow \infty$ for the system on \mathbb{Z}^d , $d \geq 3$, $\log t$ for the system on \mathbb{Z}^2 . The coefficient of the leading term for each interface is also identified.

1. INTRODUCTION

Under the coexistence of distinct phases, interfaces separating these phases are formed. One of problems on the interface separating phases is the study of the effect of a hard wall. Once imposing the effect of the hard wall on random interfaces, interfaces are pushed up by the fluctuation. The problem to identify how high the interface is pushed up is called entropic repulsion.

The paper [4] investigated the such problem in the dynamical situation and identified the asymptotic behavior of the height for the Ginzburg-Landau $\nabla\phi$ interface model on the hard wall. This dynamics can be regarded as the motion of the interface separating two phases and reflected by the hard wall. In this paper, let us discuss the entropic repulsion for two interfaces separating three distinct phases, which are reflected by the hard wall and interacting with one another by the exclusion. We shall consider the stochastic interface $\Phi_t = \{(\phi_t^{(1)}(x), \phi_t^{(2)}(x)); x \in \mathbb{Z}^d\}$ governed by the following system of SDEs of Skorokhod type:

$$(1.1) \quad \begin{cases} d\phi_t^{(1)}(x) = \Delta\phi_t^{(1)}(x)dt + \sqrt{2}dw_t^{(1)}(x) + d\ell_t^{(1)}(x) - d\ell_t^{(2)}(x), \\ d\phi_t^{(2)}(x) = \Delta\phi_t^{(2)}(x)dt + \sqrt{2}dw_t^{(2)}(x) + d\ell_t^{(2)}(x), \\ \ell_t^{(1)}(x) \text{ and } \ell_t^{(2)}(x) \text{ are increasing in } t, \\ 0 \leq \phi_t^{(1)}(x) \leq \phi_t^{(2)}(x), \quad t \geq 0, \\ \int_0^\infty \phi_t^{(1)}(x) d\ell_t^{(1)}(x) = 0, \\ \int_0^\infty (\phi_t^{(2)}(x) - \phi_t^{(1)}(x)) d\ell_t^{(2)}(x) = 0, \end{cases}$$

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for $x \in \mathbb{Z}^d$, where the operator Δ is the discrete Laplacian on \mathbb{Z}^d , that is,

$$\Delta\zeta(x) = \sum_{\substack{y \in \mathbb{Z}^d, \\ |x-y|=1}} (\zeta(y) - \zeta(x))$$

for $\zeta = \{\zeta(x) \in \mathbb{R}; x \in \mathbb{Z}^d\} \in \mathbb{R}^{\mathbb{Z}^d}$. Here, $\{w_t^{(i)}(x); x \in \mathbb{Z}^d, i = 1, 2\}$ is the family of independent one-dimensional standard Brownian motions. The processes $\ell_t^{(1)}(x)$ and $\ell_t^{(2)}(x)$ are called the local time for $\phi_t^{(1)}(x)$ and $\phi_t^{(2)}(x) - \phi_t^{(1)}(x)$, respectively.

We assume the following condition on the initial data throughout this paper.

Assumption 1. We assume that the initial data of Φ_t is distributed by a independent identical distribution $\prod_{x \in \mathbb{Z}^d} \rho(d\phi^{(1)}(x), d\phi^{(2)}(x))$. We moreover assume that ρ satisfies the following condition:

- (i) the support of ρ is in $\{(u, v) \in \mathbb{R}^2; 0 \leq u \leq v\}$,
- (ii) ρ has a finite second moment,
- (iii) ρ is absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^2 .

The conditions (i) and (ii) is quite natural for the existence of the solution of (1.1). The condition (iii) is rather technical requirement. It is imposed in order to calculate the relative entropy, see Section 3. We now state our main result which gives the precise asymptotic behavior of height of interfaces.

Theorem 1.1. *We have for $i = 1, 2$*

$$(1.2) \quad \lim_{t \rightarrow \infty} P \left(\phi_t^{(i)}(0) \in \left(\sqrt{(C_i/2 - \epsilon) \log_d(t)}, \sqrt{(C_i/2 + \epsilon) \log_d(t)} \right) \right) = 1$$

with two constant $C_1, C_2 > 0$, where $\log_d(t)$ is defined by

$$\log_d(t) = \begin{cases} (\log t)^2, & d = 2, \\ \log t, & d \geq 3. \end{cases}$$

Here, the constants C_1, C_2 are explicitly identified as

$$C_1 = \begin{cases} \int_0^\infty p_t(0, 0) dt, & d \geq 3, \\ \lim_{t \rightarrow \infty} \frac{1}{\log t} \int_0^t p_s(0, 0) ds, & d = 2, \end{cases}$$

$$C_2 = (\sqrt{2} + 1)^2 C_1,$$

where $\{p_t(x, y)\}$ is the transition kernel of the simple random walk on \mathbb{Z}^d generated by 2Δ .

This result says that the lower interface grows up like $\sqrt{C_1 \log_d(t)/2}$ as $t \rightarrow \infty$ and the upper grows up like $(\sqrt{2} + 1)\sqrt{C_1 \log_d(t)/2}$. We note that the gap between the lower and upper grows up like $\sqrt{C_1 \log_d(t)}$ and it is different from asymptotics of the single interface, see [4]. From this fact, one can see that the upper interface is pushed up more by the exclusion from another interface than the exclusion from the hard wall.

We need to remark that this result is related to the entropic repulsion of the finite volume Gibbs measures on $\Lambda_N := [-N, N]^d \cap \mathbb{Z}^d$ of multi-layered system. It is introduced by $\mu_{\Lambda_N}^+(\cdot) = \mu_{\Lambda_N}(\cdot | 0 \leq \phi^{(1)} \leq \phi^{(2)})$, where μ_{Λ_N} is given by

$$\begin{aligned} \mu_{\Lambda_N}(d\phi^{(1)}d\phi^{(2)}) &= \frac{1}{Z_{\Lambda_N}} \exp \left(\sum_{i=1,2} \sum_{x \in \Lambda} \phi^{(i)}(x) \Delta \phi^{(i)}(x) \right) \\ &\quad \times \prod_{x \in \Lambda_N} d\phi^{(1)}(x) d\phi^{(2)}(x) \prod_{x \in \mathbb{Z}^d \setminus \Lambda_N} \delta_0(d\phi^{(1)}(x)) \delta_0(d\phi^{(2)}(x)) \end{aligned}$$

where Z_{Λ_N} is the normalizing constant. In [1] (when $d \geq 3$) and [8] (when $d = 2$), the asymptotic behavior of sample mean is identified as follows:

$$\lim_{N \rightarrow \infty} \mu_{\Lambda_N} \left(\left| \Lambda_{\kappa N} \right|^{-1} \sum_{x \in \Lambda_{\kappa N}} \phi^{(i)}(x) \in \left(\sqrt{(C_i - \epsilon) \log_d(N)}, \sqrt{(C_i + \epsilon) \log_d(N)} \right) \right) = 1$$

holds for every $i = 1, 2$, $\eta > 0$ and $0 < \kappa < 1$. Here, the constants C_1, C_2 is same as in Theorem 1.1. We can easily to see that the asymptotic behavior stated above coincides with (1.2) with taking $t = N^{2 \pm \epsilon}$.

Remark 1.1. We require the asymptotic behavior stated above to show Theorem 1.1. Since such asymptotic behavior of the height of the interface distributed by the Gibbs measure is unknown in general, we only think of the Gaussian system (1.1).

We shall prove Theorem 1.1 by a similar method to [4]. We split the proof of Theorem 1.1 into two parts, the lower bound and the upper bound of the height of interfaces, see Theorems 3.1 and 4.1, respectively. For the lower bound, we derive it by reducing our problem to that for the finite volume Gibbs measure with applying the comparison theorem and the logarithmic Sobolev inequality. Though we have the pointwise estimate for the finite volume Gibbs measure in the case of the single interface (see [2]), we have only the estimate for the sample mean, see [1] and [8] for details, and it is not enough for our goal. However, since we can show the lower bound for the expectation value with the help of the result of [1] and [8], we shall establish the estimate for the variance and show that the order of the fluctuation is relatively small to that of the expectation value, see Proposition 3.3 for details. The proof of the upper bound is rather simple. We construct the suitable dynamics which always stays above (1.1) and give the upper bound for the introduced with applying the result of [4].

The organization of this paper is the following. In Section 2, we study several properties of (1.1) and the dynamics on a finite set corresponding to (1.1). In Section 3 and 4, we shall give the lower bound and the upper bound of the height of the interface, respectively.

2. DYNAMICS ON AN INFINITE SET AND ON A FINITE SET

2.1. Notations. Before starting the discussion, we shall prepare several notations which are used later.

Let $(\mathbb{Z}^d)^*$ be the set of all directed bonds $b = (x, y)$, $x, y \in \mathbb{Z}^d$, $|x - y| = 1$ in \mathbb{Z}^d . We write $x_b = x$ and $y_b = y$ for $b = (x, y)$. For every subset Λ of \mathbb{Z}^d , we denote the set of all

directed bonds included Λ and touching Λ by Λ^* and $\overline{\Lambda^*}$, respectively. That is,

$$\begin{aligned}\Lambda^* &:= \{b \in (\mathbb{Z}^d)^*; x_b \in \Lambda \text{ and } y_b \in \Lambda\}, \\ \overline{\Lambda^*} &:= \{b \in (\mathbb{Z}^d)^*; x_b \in \Lambda \text{ or } y_b \in \Lambda\}.\end{aligned}$$

For a height variable $\phi = \{\phi(x); x \in \mathbb{Z}^d\} \in \mathbb{R}^{\mathbb{Z}^d}$, we define the (discrete) gradient operator ∇ by $\nabla\phi(b) := \phi(x) - \phi(y)$ for $\phi \in \mathbb{R}^{\mathbb{Z}^d}$ and $b = (x, y) \in (\mathbb{Z}^d)^*$. Here, we note that the discrete Laplacian has the following expression:

$$\nabla\phi(x) = - \sum_{b \in (\mathbb{Z}^d)^*: x_b = x} \nabla\phi(b), \quad \phi \in \mathbb{R}^{\mathbb{Z}^d}.$$

We denote $\phi \leq \psi$ for $\phi, \psi \in \mathbb{R}^{\mathbb{Z}^d}$, if $\phi(x) \leq \psi(x)$ holds for every $x \in \mathbb{Z}^d$. Similarly, for $\Phi = (\phi^{(1)}, \phi^{(2)})$, $\Psi = (\psi^{(1)}, \psi^{(2)}) \in \mathbb{R}^{\mathbb{Z}^d} \times \mathbb{R}^{\mathbb{Z}^d}$, we denote $\Phi \leq \Psi$ if $\phi^{(1)} \leq \psi^{(1)}$ and $\phi^{(2)} \leq \psi^{(2)}$ hold. We denote the configuration $\phi \in \mathbb{R}^{\mathbb{Z}^d}$ such that $\phi(x) = 0$ for every $x \in \mathbb{Z}^d$ simply by 0.

Let us define a space of height \mathcal{X}_r^2 ($r > 0$) by

$$\mathcal{X}_r^2 = \left\{ \Phi = (\phi^{(1)}, \phi^{(2)}) \in \mathbb{R}^{\mathbb{Z}^d} \times \mathbb{R}^{\mathbb{Z}^d}; \|\phi\|_r^2 := \sum_{i=1,2} \sum_{x \in \mathbb{Z}^d} |\phi^{(i)}(x)|^2 \exp(-2r|x|) < \infty \right\}.$$

We note that \mathcal{X}_r^2 is Hilbert space with the inner product

$$(\Phi, \Psi)_r = \sum_{i=1,2} \sum_{x \in \mathbb{Z}^d} \phi^{(i)}(x) \psi^{(i)}(x) \exp(-2r|x|)$$

for $\Phi = (\phi^{(1)}, \phi^{(2)})$, $\Psi = (\psi^{(1)}, \psi^{(2)}) \in \mathbb{R}^{\mathbb{Z}^d} \times \mathbb{R}^{\mathbb{Z}^d}$. We define the space of height variables $\mathcal{X}_{r,+}^2$ ($r > 0$) by $\mathcal{X}_{r,+}^2 = \{\Phi = (\phi^{(1)}, \phi^{(2)}) \in \mathcal{X}_r^2; 0 \leq \phi^{(1)} \leq \phi^{(2)}\}$, which is the actual state space of the solution Φ_t of (1.1).

For $\Phi = (\phi^{(1)}, \phi^{(2)}) \in \mathbb{R}^{\mathbb{Z}^d} \times \mathbb{R}^{\mathbb{Z}^d}$, we say “ Φ has a compact support” when there exists a finite $\Lambda \subset \mathbb{Z}^d$ such that

$$\phi^{(i)}(x) = 0, \quad x \in \mathbb{Z}^d \setminus \Lambda, i = 1, 2$$

and we then denote the smallest Λ by $\text{supp } \Phi$.

2.2. The dynamics on a finite set. We introduce the dynamics $\Phi_t^\Lambda = (\phi_t^{(1),\Lambda}, \phi_t^{(2),\Lambda})$ on finite Λ by the following system of SDEs of Skorokhod type:

$$(2.1) \quad \begin{cases} d\phi_t^{(1),\Lambda}(x) = \Delta\phi_t^{(1),\Lambda}(x)dt + \sqrt{2}dw_t^{(1)}(x) + d\ell_t^{(1),\Lambda}(x) - d\ell_t^{(2),\Lambda}(x), & x \in \Lambda, \\ d\phi_t^{(2),\Lambda}(x) = \Delta\phi_t^{(2),\Lambda}(x)dt + \sqrt{2}dw_t^{(2)}(x) + d\ell_t^{(2),\Lambda}(x), & x \in \Lambda, \\ \phi_t^{(1),\Lambda}(x) = \phi_0^{(1),\Lambda}(x), & x \in \mathbb{Z}^d \setminus \Lambda \\ \phi_t^{(2),\Lambda}(x) = \phi_0^{(2),\Lambda}(x), & x \in \mathbb{Z}^d \setminus \Lambda \\ 0 \leq \phi_t^{(1)}(x) \leq \phi_t^{(2)}(x), & t \geq 0, x \in \mathbb{Z}^d \\ \ell_t^{(1)}(x) \text{ and } \ell_t^{(2)}(x) \text{ are increasing in } t, \\ \int_0^\infty \phi_t^{(1)}(x) d\ell_t^{(1)}(x) = 0, \\ \int_0^\infty (\phi_t^{(2)}(x) - \phi_t^{(1)}(x)) d\ell_t^{(2)}(x) = 0. \end{cases}$$

There exists the unique strong solution $\Phi_t^\Lambda = (\phi_t^{(1),\Lambda}, \phi_t^{(2),\Lambda})$ of (1.1) with an arbitrary initial data satisfying $0 \leq \phi_0^{(1)} \leq \phi_0^{(2)}$. When $\Lambda = \Lambda_N := [-N, N]^d \cap \mathbb{Z}^d$, we simply denote $\Phi_t^{\Lambda_N}(x)$, $\phi_t^{(i),\Lambda_N}(x)$ and $\ell_t^{(i),\Lambda_N}(x)$ by $\Phi_t^N(x)$, $\phi_t^{(i),N}(x)$ and $\ell_t^{(i),N}(x)$, respectively.

We should note that the comparison theorem holds for the dynamics (1.1). It will help us to show the lower bound.

Proposition 2.1. *Let $\Phi_t = (\phi_t^{(1)}, \phi_t^{(2)})$ and $\Phi_t^\Lambda = (\phi_t^{(1),\Lambda}, \phi_t^{(2),\Lambda})$ be solutions of (1.1) and (2.1) with common Brownian motions $\{w_t^{(i)}(x); x \in \mathbb{Z}^d, i = 1, 2\}$. We assume that Φ_0^Λ and Φ_0 satisfy $\phi_0^{(i),\Lambda}(x) = \phi_0^{(i)}(x)$ for $x \in \Lambda$ and $i = 1, 2$ and $\phi_0^{(i),\Lambda}(x) = 0$ for $x \in \mathbb{Z}^d \setminus \Lambda$. Then, $\Phi_t^\Lambda \leq \Phi_t$ holds for every $t \geq 0$.*

Proof. Since $0 = \phi_t^{(i),\Lambda}(x) \leq \phi_t^{(i)}(x)$ holds outside of Λ , it is sufficient to show $\phi_t^{(i),\Lambda}(x) \leq \phi_t^{(i)}(x)$ for $x \in \Lambda$. We define $\psi^{(i)}(x) = \phi_t^{(i),\Lambda}(x) - \phi_t^{(i)}(x)$ for $x \in \mathbb{Z}^d$ and $i = 1, 2$. We then have

$$\begin{aligned} & \sum_{i=1,2} \sum_{x \in \Lambda} \left(\psi_t^{(i)}(x)^+ \right)^2 \\ & \leq 2 \sum_{i=1,2} \sum_{x \in \Lambda} \int_0^t \psi_s^{(i)}(x)^+ \Delta \psi_s^{(i)}(x) ds \\ & \quad + 2 \sum_{x \in \Lambda} \int_0^t \psi_s^{(i)}(x)^+ \left(d\ell_t^{(1),\Lambda}(x) - d\ell_t^{(1)}(x) \right) \\ & \quad + 2 \sum_{x \in \Lambda} \int_0^t \left(\psi_s^{(2)}(x)^+ - \psi_s^{(1)}(x)^+ \right) \left(d\ell_t^{(2),\Lambda}(x) - d\ell_t^{(2)}(x) \right) \\ & =: F_1 + F_2 + F_3. \end{aligned}$$

Here, by the definition of the local time $\ell_t^{(1),\Lambda}(x)$ and $\ell_t^{(1)}(x)$, we obtain the second term F_2 is nonpositive. Noting that

$$\left\{ (a-b)^+ - (c-d)^+ \right\} \left\{ (a-c)^+ - (b-d)^+ \right\} \geq 0$$

holds for every $a, b, c, d \in \mathbb{R}$, the sign of $\psi_s^{(2)}(x)^+ - \psi_s^{(1)}(x)^+$ is same as of

$$\left(\phi_s^{(2),\Lambda}(x) - \phi_s^{(1),\Lambda}(x) \right)^+ - \left(\phi_s^{(2)}(x) - \phi_s^{(1)}(x) \right)^+.$$

We therefore obtain that the third term F_3 is also nonpositive. Using the summation-by-parts formula

$$(2.2) \quad \sum_{x \in \Lambda} \phi(x) \Delta \psi(x) = - \sum_{b \in \Lambda^*} \nabla \phi(b) \nabla \psi(b)$$

for every $\phi, \psi \in \mathbb{R}^{\mathbb{Z}^d}$ such that $\phi(x) = 0$ on $\mathbb{Z}^d \setminus \Lambda$ and

$$(u^+ - v^+)(u - v) \geq (u^+ - v^+)^2$$

for every $u, v \in \mathbb{R}$, we obtain that the first term F_1 is nonpositive. Summarizing above, we obtain $\psi_t^{(i)}(x) = 0$ for every $t \geq 0$ and $x \in \Lambda$, which shows the conclusion. \square

2.3. Finite volume Gibbs measures and Dirichlet forms. We introduce Gibbs measures associated with (2.1). However, there does not exist an infinite volume Gibbs measure. We therefore introduce finite volume Gibbs measures only.

We at first introduce the finite volume Gibbs measure for the case without any interaction. We define a probability measure $\nu_{\Lambda,\xi}$ on \mathbb{R}^Λ by

$$\nu_{\Lambda,\xi}(d\phi) = Z_{\Lambda,\xi}^{-1} \exp(-H(\phi \wedge \xi)) \prod_{x \in \Lambda} d\phi^+(x),$$

where $d\phi(x)$ is the Lebesgue measure on $[0, \infty)$. This measure ν is the finite volume Gibbs measure corresponding to the single interface on the wall. We define $\bar{\mu}_{\Lambda,\xi}$ by $\bar{\mu}_{\Lambda,\xi} = \nu_{\Lambda,\xi_1} \times \nu_{\Lambda,\xi_2}$, which is the finite volume Gibbs measure associated with two interfaces on the wall without any interaction. We define $\mu_{\Lambda,\xi}$ by $\mu_{\Lambda,\xi} = \bar{\mu}_{\Lambda,\xi}(\cdot | \phi^{(2)} \geq \phi^{(1)})$, which is the finite volume Gibbs measure corresponding to the dynamics (2.1) with $\xi \equiv \phi_0^\Lambda$. Since we only deal with the case $\xi \equiv 0$, we simply denote $\mu_{\Lambda,0}$ by μ_Λ if no confusion arises.

We shall introduce an approximating sequence for μ_Λ and the equation (2.1). Let $\rho \in C_0^\infty(\mathbb{R})$ be non-negative, symmetric ($\rho(u) = \rho(-u)$), $\int_{\mathbb{R}} \rho(u) du = 1$ and $\rho(u) = 0$ if $|u| \geq 1$. For $0 < \delta \leq 1$, we define ρ_δ by $\rho_\delta(u) := \delta^{-1} \rho(\delta^{-1}u)$ for $u \in \mathbb{R}$. We define χ_α and $\chi_{\alpha,\delta}$ by $\chi_\alpha(u) = \alpha^{-1}(u^-)^2/2$ and $\chi_{\alpha,\delta}(u) = (\rho_\delta * \chi_\alpha(\cdot + \delta))(u)$, respectively. For $\epsilon = (\epsilon_1, \epsilon_2)$, $\epsilon_1, \epsilon_2 > 0$ and $\delta > 0$, we define the probability measure $\mu_{\Lambda,\epsilon}$ and $\mu_{\Lambda,\epsilon,\delta}$ on $(\mathbb{R}^\Lambda)^2$ by

$$\frac{d\mu_{\Lambda,\epsilon}}{d\bar{\mu}_\Lambda} = Z_{\Lambda,\epsilon}^{-1} \exp \left(- \sum_{x \in \Lambda} W_\epsilon(\phi^{(1)}(x), \phi^{(2)}(x)) \right),$$

and

$$\frac{d\mu_{\Lambda,\epsilon,\delta}}{d\bar{\mu}_\Lambda} = Z_{\Lambda,\epsilon,\delta}^{-1} \exp \left(- \sum_{x \in \Lambda} W_{\epsilon,\delta}(\phi^{(1)}(x), \phi^{(2)}(x)) \right),$$

respectively, where W_ϵ and $W_{\epsilon,\delta}$ is defined by

$$\begin{aligned} W_\epsilon(u, v) &= \chi_{\epsilon_1}(u) + \chi_{\epsilon_2}(v - u), \\ W_{\epsilon,\delta}(u, v) &= \chi_{\epsilon_1,\delta}(u) + \chi_{\epsilon_2,\delta}(v - u) \end{aligned}$$

for $u, v \in \mathbb{R}$. Here, $Z_{\Lambda,\epsilon}$ and $Z_{\Lambda,\epsilon,\delta}$ are normalizing constants. We introduce the Dirichlet form $\mathcal{E}_\epsilon^\Lambda$ and $\mathcal{E}_{\epsilon,\delta}^\Lambda$ by

$$\mathcal{E}_\epsilon^\Lambda(F, G) = \sum_{i=1,2} \sum_{x \in \Lambda} \int_{(\mathbb{R}^\Lambda)^2} \frac{\partial F}{\partial \phi^{(i)}(x)} \frac{\partial G}{\partial \phi^{(i)}(x)} d\mu_\epsilon(d\phi)$$

and

$$\mathcal{E}_{\epsilon,\delta}^\Lambda(F, G) = \sum_{i=1,2} \sum_{x \in \Lambda} \int_{(\mathbb{R}^\Lambda)^2} \frac{\partial F}{\partial \phi^{(i)}(x)} \frac{\partial G}{\partial \phi^{(i)}(x)} d\mu_{\epsilon,\delta}(d\phi)$$

for $F, G \in C^2((\mathbb{R}^\Lambda)^2)$, respectively. We note that $\mathcal{E}_\epsilon^\Lambda$ is the Dirichlet form associated to SDEs

$$(2.3) \quad \begin{cases} d\phi_t^{(1),\Lambda,\epsilon}(x) = \Delta \phi_t^{(1),\Lambda,\epsilon}(x) dt + \sqrt{2} dw_t^{(1)}(x) + d\ell_t^{(1),\Lambda,\epsilon}(x) - d\ell_t^{(2),\Lambda,\epsilon}(x), \\ d\phi_t^{(2),\Lambda,\epsilon}(x) = \Delta \phi_t^{(2),\Lambda,\epsilon}(x) dt + \sqrt{2} dw_t^{(2)}(x) + d\ell_t^{(2),\Lambda,\epsilon}(x) \end{cases}$$

for $x \in \Lambda$, where $\ell_t^{(1),\epsilon}(x)$ $\ell_t^{(2),\epsilon}(x)$ are defined by

$$\begin{aligned}\ell_t^{(1),\Lambda,\epsilon}(x) &= \epsilon_1^{-1} \int_0^t \phi_s^{(1),\Lambda,\epsilon}(x)^- ds, \\ \ell_t^{(2),\Lambda,\epsilon}(x) &= \epsilon_2^{-1} \int_0^t (\phi_s^{(2),\Lambda,\epsilon}(x) - \phi_s^{(1),\Lambda,\epsilon}(x))^- ds.\end{aligned}$$

We also note that $\mathcal{E}_{\epsilon,\delta}^\Lambda$ is the Dirichlet form associated to SDEs

$$(2.4) \quad \begin{cases} d\phi_t^{(1),\Lambda,\epsilon,\delta}(x) = \Delta\phi_t^{(1),\Lambda,\epsilon,\delta}(x) dt + \sqrt{2}dw_t^{(1)}(x) + d\ell_t^{(1),\Lambda,\epsilon,\delta}(x) - d\ell_t^{(2),\Lambda,\epsilon,\delta}(x), \\ d\phi_t^{(2),\Lambda,\epsilon,\delta}(x) = \Delta\phi_t^{(2),\Lambda,\epsilon,\delta}(x) dt + \sqrt{2}dw_t^{(2)}(x) + d\ell_t^{(2),\Lambda,\epsilon,\delta}(x) \end{cases}$$

for $x \in \Lambda$, where $\ell_t^{(1),\epsilon}(x)$ $\ell_t^{(2),\epsilon}(x)$ are defined by

$$\begin{aligned}\ell_t^{(1),\Lambda,\epsilon}(x) &= - \int_0^t \chi'_{\epsilon_1,\delta}(\phi_s^{(1),\Lambda,\epsilon,\delta}(x)) ds, \\ \ell_t^{(2),\Lambda,\epsilon}(x) &= - \int_0^t \chi'_{\epsilon_2,\delta}(\phi_s^{(2),\Lambda,\epsilon,\delta}(x) - \phi_s^{(1),\Lambda,\epsilon,\delta}(x)) ds.\end{aligned}$$

It is easy to show that there exist unique strong solutions for (2.3) and (2.4), respectively, since all coefficients are Lipschitz continuous. Noting that $W_{\epsilon,\delta}$ converges to W_ϵ uniformly in \mathbb{R}^2 for fixed $\epsilon = (\epsilon_1, \epsilon_2)$, $\epsilon_1, \epsilon_2 > 0$, it is easy to see the following identity:

$$(2.5) \quad \lim_{\delta \downarrow 0} E \left[\sum_{x \in \Lambda} \sum_{i=1,2} \left| \phi_t^{(i),\Lambda,\epsilon,\delta}(x) - \phi_t^{(i),\Lambda,\epsilon}(x) \right|^2 \right] = 0$$

for every $t \geq 0$. We can also show that $\Phi_t^{\Lambda,\epsilon}$ converges to Φ_t^Λ as $\epsilon_1 \downarrow 0$ and $\epsilon_2 \downarrow 0$ in the following sense:

$$(2.6) \quad \lim_{\epsilon_2 \downarrow 0} \lim_{\epsilon_1 \downarrow 0} E \left[\sum_{x \in \Lambda} \sum_{i=1,2} \left| \phi_t^{(i),\Lambda}(x) - \phi_t^{(i),\Lambda,\epsilon}(x) \right|^2 \right] = 0$$

for every $t > 0$ and $\Lambda \subset \mathbb{Z}^d$, see Theorem 2.3 for details. Furthermore, we can show

$$(2.7) \quad \lim_{\Lambda \uparrow \mathbb{Z}^d} E \left[\sum_{i=1,2} \left| \phi_t^{(i)}(x) - \phi_t^{(i),\Lambda}(x) \right|^2 \right] = 0$$

for every $x \in \mathbb{Z}^d$ and $t > 0$, which will be claimed in Theorem 2.7. These identities imply that the estimate for the variance of ϕ_t can be reduced to that for $\phi_t^{\Lambda,\epsilon,\delta}$. We shall discuss an important property of $\phi_t^{\Lambda,\epsilon,\delta}$, comparison theorem with respect to initial datum. This plays the key role in the proof of Theorem 3.3.

Proposition 2.2. *Let $\Phi_t^{\Lambda,\epsilon,\delta}$ and $\tilde{\Phi}_t^{\Lambda,\epsilon,\delta}$ be the solution of (2.4) with initial data satisfying $\Phi_0^{\Lambda,\epsilon,\delta} \leq \tilde{\Phi}_0^{\Lambda,\epsilon,\delta}$. We then have $\Phi_t^{\Lambda,\epsilon,\delta} \leq \tilde{\Phi}_t^{\Lambda,\epsilon,\delta}$ and*

$$(2.8) \quad \sum_{i=1,2} \left(\tilde{\phi}_t^{(i),\Lambda,\epsilon,\delta}(x) - \phi_t^{(i),\Lambda,\epsilon,\delta}(x) \right) \leq \sum_{y \in \Lambda} \sum_{i=1,2} p_t(x, y) \left(\tilde{\phi}_0^{(i),\Lambda,\epsilon,\delta}(y) - \phi_0^{(i),\Lambda,\epsilon,\delta}(y) \right)$$

for every $t > 0$ and $x \in \Lambda$.

Proof. We shall at first show the first inequality in (2.8). We define $\psi^{(i)}$ by $\psi_t^{(i)} = \tilde{\phi}_t^{(i),\Lambda,\epsilon,\delta} - \phi_t^{(i),\Lambda,\epsilon,\delta}$. Differentiating $\sum_{i=1,2} \sum_{x \in \Lambda} (\psi_t^{(i)}(x)^-)^2$ in t and integrating in t , we obtain

$$\begin{aligned}
& \sum_{i=1,2} \sum_{x \in \Lambda} \left(\psi_t^{(i)}(x)^- \right)^2 \\
& \leq -2 \sum_{i=1,2} \sum_{x \in \Lambda} \int_0^t \psi_s^{(i)}(x)^- \Delta \psi_s^{(i)}(x) ds \\
& \quad - 2 \sum_{x \in \Lambda} \int_0^t \psi_s^{(1)}(x)^- \left(\chi'_{\epsilon_1, \delta}(\phi_s^{(1), \Lambda, \epsilon, \delta}(x)) - \chi'_{\epsilon_1, \delta}(\tilde{\phi}_s^{(1), \Lambda, \epsilon, \delta}(x)) \right) ds \\
& \quad - 2 \sum_{x \in \Lambda} \int_0^t \left(\psi_s^{(2)}(x)^- - \psi_s^{(1)}(x)^- \right) \\
& \quad \quad \times \left(\chi'_{\epsilon_2, \delta}(\phi_s^{(2), \Lambda, \epsilon, \delta}(x) - \phi_s^{(1), \Lambda, \epsilon, \delta}(x)) - \chi'_{\epsilon_2, \delta}(\tilde{\phi}_s^{(2), \Lambda, \epsilon, \delta}(x) - \tilde{\phi}_s^{(1), \Lambda, \epsilon, \delta}(x)) \right) ds \\
& =: F_1 + F_2 + F_3.
\end{aligned}$$

Noting that $\chi'_{\epsilon, \delta}(u)$ is nondecreasing in $u \in \mathbb{R}$ and that

$$\{(a-b)^- - (c-d)^-\} \{(a-c)^- - (b-d)^-\} \geq 0$$

holds for every $a, b, c, d \in \mathbb{R}$, we obtain that F_2 and F_3 in the right hand side are non-positive. In order to obtain the first inequality in (2.8), we only need to show that the first term F_1 is nonpositive. Using (2.2) and $(u-v)(u^- - v^-) \leq -(u^- - v^-)^2$ for every $u, v \in \mathbb{R}^d$, we get

$$\sum_{x \in \Lambda} \psi_s^{(i)}(x)^- \Delta \psi_s^{(i)}(x) \geq \frac{1}{2} \sum_{b \in \Lambda^*} \left(\nabla (\psi^{(i)})^-(b) \right)^2 \geq 0.$$

We have used $\psi_s^{(i)}(x)^- = 0$ for every $x \in \mathbb{Z}^d \setminus \Lambda$. This shows that the term F_1 is nonpositive.

We next show the inequality (2.8). Assuming $\alpha_s(x, y)$ be continuous differentiable in $s \in [0, t]$, we then obtain

$$\begin{aligned}
\sum_{i=1,2} \alpha_t(y) \psi_t^{(i)}(y) &= \sum_{i=1,2} \alpha_0(y) \psi_0^{(i)}(y) \\
&+ \sum_{i=1,2} \int_0^t \frac{d}{ds} \alpha_s(y) \psi_s^{(i)}(y) ds \\
&+ \sum_{i=1,2} \int_0^t \alpha_s(y) \Delta \psi_s^{(i)}(y) ds \\
&- \int_0^t \alpha_s(y) \left(\chi'_{\epsilon_1, \delta}(\tilde{\phi}_s^{(1), \Lambda, \epsilon}(y)) - \chi'_{\epsilon_1, \delta}(\phi_s^{(1), \Lambda, \epsilon}(y)) \right) ds
\end{aligned}$$

Note that

$$\frac{\partial W_{\epsilon, \delta}}{\partial u} + \frac{\partial W_{\epsilon, \delta}}{\partial v} = \chi'_{\epsilon_1, \delta}(u)$$

holds by definition of $W_{\epsilon,\delta}$. Since $\chi'_{\epsilon_1,\delta}(u)$ is nondecreasing in $u \in \mathbb{R}$, we obtain that the fifth terms is nonpositive. Taking the sum over $y \in \Lambda$, we get

$$\begin{aligned} \sum_{i=1,2} \sum_{x \in \Lambda} \alpha_t(y) \psi_t^{(i)}(y) &\leq \sum_{i=1,2} \sum_{x \in \Lambda} \alpha_0(y) \psi_0^{(i)}(y) \\ &\quad + \sum_{i=1,2} \sum_{x \in \Lambda} \int_0^t \frac{d}{ds} \alpha_s(y) \psi_s^{(i)}(y) ds \\ &\quad + \sum_{i=1,2} \sum_{x \in \Lambda} \int_0^t \Delta \alpha_s(x) \psi_s^{(i)}(x) ds. \end{aligned}$$

Here, we have applied (2.2) twice. Choosing $\alpha_s(y) = p_{t-s}^\Lambda(x, y)$, we obtain the second and third terms cancel out, since $\alpha_s(y)$ is the solution of the forward equation

$$\begin{cases} \frac{d}{dt} \alpha_s(y) = -\Delta \alpha_s(y), & y \in \Lambda, \\ \alpha_s(y) = 0, & y \in \mathbb{Z}^d \setminus \Lambda, t \geq 0 \\ \alpha_t(y) = \delta_x(y), & y \in \Lambda. \end{cases}$$

Using trivial inequality $p_t^\Lambda(x, y) \leq p_t(x, y)$, we get the conclusion. \square

2.4. Approximation scheme for the dynamics on the finite set. For a finite subset $\Lambda \subset \mathbb{Z}^d$, let us consider the dynamics $\Phi_t^\Lambda = (\phi_t^{(1),\Lambda}, \phi_t^{(2),\Lambda})$ governed by SDEs (2.1). In this subsection, we omit the domain Λ for simplicity of notations when no confusion arises.

Since the state space of (2.1) is $|\Lambda|$ -fold of two dimensional cones and therefore convex, we can apply Tanaka's result [9], and immediately obtain the existence and uniqueness of the strong solution for our equation. However, in order to show the lower bound, we need the nice approximation to the solution, see Section 3. We shall introduce an approximation scheme for solutions for our equation (2.1) from that for the equations with weak reflection terms. Our goal in this subsection is the following:

Theorem 2.3. *Let $\Phi_t = (\phi_t^{(1)}, \phi_t^{(2)})$ and $\Phi_t^\epsilon = (\phi_t^{(1),\epsilon}, \phi_t^{(2),\epsilon})$ be the solutions for (2.1) and (2.3) with common Brownian motions and initial datum $\Phi_0 = (\phi_0^1, \phi_0^2)$, respectively. We then have*

$$(2.9) \quad \phi_t^{(i)}(x) = \lim_{\epsilon_2 \downarrow 0} \lim_{\epsilon_1 \downarrow 0} \phi_t^{(i),\epsilon}(x)$$

holds almost surely for every $i = 1, 2, t > 0$ and $x \in \Lambda$. Furthermore, we obtain the identity (2.6).

Our first observation is on the monotonicity for Φ_t^ϵ in ϵ_1 , which plays key role in the proof of Theorem 2.3.

Lemma 2.4. *Fix $\epsilon = (\epsilon_1, \epsilon_2), \epsilon' = (\epsilon'_1, \epsilon'_2)$ such that $\epsilon_1 \geq \epsilon'_1$ and $\epsilon_2 = \epsilon'_2$. Let $\Phi_t^{\Lambda,\epsilon}$ and $\Phi_t^{\Lambda,\epsilon'}$ be solutions of (2.3) with common Brownian motions and initial datum. We then have $\Phi_t^{\Lambda,\epsilon} \leq \Phi_t^{\Lambda,\epsilon'}$ for every $t > 0$.*

Proof. The proof is quite parallel to that of Lemma 2.2. We define $\psi_t^{(i)}$ by $\psi_t^{(i)} = \phi_t^{(i),\epsilon} - \phi_t^{(i),\epsilon'}$. Differentiating $\sum_{i=1,2} \sum_{x \in \Lambda} (\psi_t^{(i)}(x)^-)^2$ in t and integrating in t , we obtain

$$\begin{aligned}
& \sum_{i=1,2} \sum_{x \in \Lambda} \left(\psi_t^{(i)}(x)^- \right)^2 \\
& \leq -2 \sum_{i=1,2} \sum_{x \in \Lambda} \int_0^t \psi_s^{(i)}(x)^- \Delta \psi_s^{(i)}(x) ds \\
& \quad - 2 \sum_{x \in \Lambda} \int_0^t \psi_s^{(1)}(x)^- \left(\epsilon_1^{-1} (\phi_s^{(1),\epsilon}(x))^- - \epsilon_1'^{-1} (\phi_s^{(1),\epsilon'}(x))^- \right) ds \\
& \quad - 2 \sum_{x \in \Lambda} \int_0^t (\psi_s^{(2)}(x)^- - \psi_s^{(1)}(x)^-) \\
& \quad \quad \times \left(\epsilon_2^{-1} (\phi_s^{(2),\Lambda,\epsilon,\delta}(x) - \phi_s^{(1),\Lambda,\epsilon,\delta}(x))^- - \epsilon_2^{-1} (\phi_s^{(2),\epsilon'}(x) - \phi_s^{(1),\epsilon'}(x))^- \right) ds \\
& =: F_1 + F_2 + F_3.
\end{aligned}$$

The same observation as in the proof of Lemma 2.2 shows that F_1 and F_3 are nonpositive. Here, since we have assumed $\epsilon_1 \geq \epsilon_1'$, we obtain

$$F_2 \leq -2 \sum_{x \in \Lambda} \int_0^t \psi_s^{(1)}(x)^- \left(\epsilon_1^{-1} (\phi_s^{(1),\epsilon}(x))^- - \epsilon_1'^{-1} (\phi_s^{(1),\epsilon'}(x))^- \right) ds.$$

Using the same observation as in the proof of Lemma 2.2 again, we obtain that F_2 is also nonpositive. Summarizing the above, we finally get

$$\sum_{i=1,2} \sum_{x \in \Lambda} (\psi_t^{(i)}(x)^-)^2 = 0$$

for every $t \geq 0$, which indicates the conclusion. \square

Lemma 2.4 guarantees that there exists a limit $\Phi_t^{(0,\epsilon_2)}$ of Φ_t^ϵ as $\epsilon_1 \downarrow 0$. Though we want to take the limit as $\epsilon_2 \downarrow 0$ also, we can not expect the monotonicity in ϵ_2 as in Lemma 2.4. In order to establish the monotonicity in some sense, let us transform the solution Φ_t^ϵ as follows: we define $\Psi_t^\epsilon = (\psi_t^{(1),\epsilon}, \psi_t^{(2),\epsilon}) \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$ by

$$\begin{aligned}
\psi_t^{(1),\epsilon}(x) &= \frac{1}{\sqrt{2}} \left(\phi_t^{(1),\epsilon}(x) + \phi_t^{(2),\epsilon}(x) \right) \\
\psi_t^{(2),\epsilon}(x) &= \frac{1}{\sqrt{2}} \left(-\phi_t^{(1),\epsilon}(x) + \phi_t^{(2),\epsilon}(x) \right).
\end{aligned}$$

for every $x \in \Lambda$ and $t \geq 0$, that is, $\Psi_t^\epsilon(x) = (\psi_t^{(1),\epsilon}(x), \psi_t^{(2),\epsilon}(x))$ is the rotation of $\Phi_t^\epsilon(x) = (\phi_t^{(1),\epsilon}(x), \phi_t^{(2),\epsilon}(x))$ clockwise through 45 degrees. One can easily see that Ψ_t^ϵ solves SDEs (2.10)

$$\begin{cases} d\psi_t^{(1),\epsilon}(x) = \Delta \psi_t^{(1),\epsilon} dt + \sqrt{2} d\hat{w}_t^{(1)} + d\hat{\ell}_t^{(1),\epsilon}(x), & x \in \Lambda, \\ d\psi_t^{(2),\epsilon}(x) = \Delta \psi_t^{(2),\epsilon} dt + \sqrt{2} d\hat{w}_t^{(2)} + d\hat{\ell}_t^{(2),\epsilon}(x) - d\hat{\ell}_t^{(1),\epsilon}(x), & x \in \Lambda, \\ \psi_t^{(i),\epsilon}(x) \equiv \psi_0^{(i),\epsilon}(x), & x \in \mathbb{Z}^d \setminus \Lambda, i = 1, 2, \end{cases}$$

where $\hat{\ell}_t^{(1),\epsilon}(x)$ and $\hat{\ell}_t^{(2),\epsilon}(x)$ are defined by

$$(2.11) \quad \hat{\ell}_t^{(1),\epsilon}(x) = \frac{1}{2}\epsilon_1^{-1} \int_0^t (\psi_s^{(1),\epsilon}(x) - \psi_s^{(2),\epsilon}(x))^- ds,$$

$$(2.12) \quad \hat{\ell}_t^{(2),\epsilon}(x) = 2\epsilon_2^{-1} \int_0^t (\psi_s^{(2),\epsilon}(x))^- ds.$$

Here, $\hat{w}_t(x) = (\hat{w}_t^{(1)}(x), \hat{w}_t^{(2)}(x))$ is also the rotation of $w_t(x) = (w_t^{(1)}(x), \phi_t^{(2)}(x))$ clockwise through 45 degrees, and thus a two-dimensional standard Brownian motion again. We can show the monotonicity for Ψ_t^ϵ in ϵ_2 quite parallel to Lemma 2.4.

Lemma 2.5. *Fix $\epsilon = (\epsilon_1, \epsilon_2), \epsilon' = (\epsilon'_1, \epsilon'_2)$ such that $\epsilon_1 = \epsilon'_1$ and $\epsilon_2 \geq \epsilon'_2$. Let $\Psi_t^{\Lambda, \epsilon}$ and $\Psi_t^{\Lambda, \epsilon'}$ be solutions of (2.10) with common Brownian motions and initial datum. We then have $\Psi_t^{\Lambda, \epsilon} \leq \Psi_t^{\Lambda, \epsilon'}$ for every $t > 0$.*

Here, Lemmas 2.4 and 2.5 imply the existence of the limit in the right hand side of (2.9) for each $t \geq 0$ and $x \in \Lambda$. Once we have the boundedness and the equicontinuity of solutions Φ_t^ϵ uniformly in ϵ , we can conclude (2.9) by the similar way to the proof of Theorem 2.1 of [4]. We also obtain (2.6) as a simple application of the monotone convergence theorem. As the final step for the proof of Theorem 2.3, let us establish bounds corresponding to them.

Lemma 2.6. *Let Φ_t^ϵ be the solution of (2.3) with initial data Φ_0^ϵ . Then, for every $T > 0$, there exists a constant $C > 0$ independent of $\epsilon_1, \epsilon_2 > 0$ such that*

$$(2.13) \quad E \left[\sup_{0 \leq t \leq T} \|\Phi_t^\epsilon\|^4 \right] \leq C(1 + \|\Phi_0^\epsilon\|^4),$$

$$(2.14) \quad E \left[\sup_{t_1 \leq t \leq t_2} \|\Phi_t^\epsilon - \Phi_{t_1}^\epsilon\|^4 \right] \leq C(1 + \|\Phi_0^\epsilon\|^4) ((t_2 - t_1)^2 + (t_2 - t_1)^3),$$

holds for $0 \leq t_1 \leq t_2 \leq T$, where $\|\Phi\|$ denotes the ordinary Euclidean norm, that is,

$$\|\Phi\|^2 = \sum_{i=1,2} \sum_{x \in \Lambda} |\phi^{(i)}(x)|^2$$

for $\Phi = (\phi^{(1)}, \phi^{(2)}) \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$.

Proof. For a fixed $\Psi = (\psi^{(1)}, \psi^{(2)}) \in \mathbb{R}^\Lambda \times \mathbb{R}^\Lambda$, we have

$$\begin{aligned} \|\Phi_t^\epsilon - \Psi\|^2 &= \|\Phi_0^\epsilon - \Psi\|^2 \\ &+ 2 \sum_{i=1,2} \sum_{x \in \Lambda} \int_0^t (\phi_s^{(i),\epsilon}(x) - \psi^{(i)}(x)) \Delta \phi_s^{(i),\epsilon}(x) ds \\ &+ 2 \sum_{x \in \Lambda} \int_0^t (\phi_s^{(1),\epsilon}(x) - \psi^{(1)}(x)) d\ell_s^{(1),\epsilon}(x) \\ &+ 2 \sum_{x \in \Lambda} \int_0^t (\phi_s^{(2),\epsilon}(x) - \phi_s^{(1),\epsilon}(x) - \psi^{(2)}(x) + \psi^{(1)}(x)) d\ell_s^{(2),\epsilon}(x) \\ &+ 2t|\Lambda| + m_t(\Psi), \end{aligned}$$

where $m_t(\Psi)$ is the martingale with the following expression:

$$m_t(\Psi) = 2 \sum_{i=1,2} \sum_{x \in \Lambda} \int_0^t (\phi_s^{(i),\epsilon}(x) - \psi^{(i)}(x)) dw_s^{(i)}(x).$$

Here, since the third term in the right hand side is expressed by

$$\sum_{x \in \Lambda} \int_0^t \phi_s^{(1),\epsilon}(x) d\ell_s^{(1),\epsilon}(x) - \sum_{x \in \Lambda} \psi^{(1)}(x) \ell_t^{(1),\epsilon}(x),$$

it is nonpositive by the definition of $\ell_s^{(1),\epsilon}(x)$. Similarly, we also obtain that the fourth term is nonpositive. Using the Schwarz inequality, we get

$$(2.15) \quad \|\Phi_t^\epsilon - \Psi\|^2 \leq \|\Phi_0^\epsilon - \Psi\|^2 + K_1 \int_0^t (1 + \|\Phi_s^\epsilon\|^2 + \|\Phi_s^\epsilon - \Psi\|^2) ds + m_t(\Psi)$$

with some constant $K_1 > 0$ independent of ϵ and $t > 0$. Repeating the argument in the proof of Lemma 2.2 of [4], we obtain (2.13) and (2.14). \square

2.5. The dynamics on the infinite lattice \mathbb{Z}^d . We shall construct the infinite system Φ_t governed by SDEs (1.1). Our goal is the following:

Theorem 2.7. *For every $\Phi_0 = (\phi_0^{(1)}, \phi_0^{(2)}) \in \mathcal{X}_{r,+}^2$, there exists an unique (strong) solution Φ_t of (1.1). Furthermore, the identity (2.7) holds.*

Let us introduce two lemmas, which imply Theorem 2.7 by applying the same argument as in the proof of Theorem 2.1 of [4]. We note that Φ_t can be obtained as the increasing limit of Φ_t^Λ , and therefore the monotone convergence theorem yields the identity (2.7).

Lemma 2.8. *Let $\Phi_t^\Lambda = (\phi_t^{(1)}, \phi_t^{(2)})$ and $\Phi_t^{\Lambda'} = (\phi_t^{(1),\Lambda'}, \phi_t^{(2),\Lambda'})$ be solutions of (2.1) with common Brownian motions $\{w_t^{(i)}(x); x \in \mathbb{Z}^d, i = 1, 2\}$ and initial datum satisfying $\text{supp } \Phi_0 \subset \Lambda$. If $\Lambda \subset \Lambda'$, then $\Phi_t^\Lambda \leq \Phi_t^{\Lambda'}$ holds.*

Lemma 2.9. *Let Φ_t^Λ be the solution of (2.1) with initial data Φ_0^Λ . Then, for every $T > 0$, there exists a constant $C > 0$ independent of Λ such that*

$$(2.16) \quad E \left[\sup_{0 \leq t \leq T} \|\Phi_t^\Lambda\|_r^4 \right] \leq C(1 + \|\Phi_0^\Lambda\|_r^4),$$

$$(2.17) \quad E \left[\sup_{t_1 \leq t \leq t_2} \|\Phi_t^\Lambda - \Phi_{t_1}^\Lambda\|_r^4 \right] \leq C(1 + \|\Phi_0^\Lambda\|_r^4)(t_2 - t_1)^2,$$

holds for $0 \leq t_1 \leq t_2 \leq T$. The estimates (2.16) and (2.17) is also true for the solution Φ_t of (1.1) if there exists.

Let us give a proof of Lemma 2.9 only, since we can show Lemma 2.8 by a similar manner to the proof of Proposition 2.1.

Proof of Lemma 2.9. For fixed $\Psi = (\psi^{(1)}, \psi^{(2)}) \in \mathcal{X}_{r,+}^2$, we have

$$\begin{aligned}
& \|\Phi_t^\Lambda - \Psi\|_r^2 \\
&= \|\Phi_0^\Lambda - \Psi\|_r^2 \\
&\quad + 2 \sum_{i=1,2} \sum_{x \in \Lambda} \exp(-2r|x|) \int_0^t (\phi_s^{(i),\Lambda}(x) - \psi^{(i)}(x)) \Delta \phi_s^{(i),\Lambda}(x) ds \\
&\quad + 2 \sum_{x \in \Lambda} \exp(-2r|x|) \int_0^t (\phi_s^{(1),\Lambda}(x) - \psi^{(1)}(x)) d\ell_s^{(1),\Lambda}(x) \\
&\quad + 2 \sum_{x \in \Lambda} \exp(-2r|x|) \int_0^t (\phi_s^{(2),\Lambda}(x) - \phi_s^{(1),\Lambda}(x) - \psi^{(2)}(x) + \psi^{(1)}(x)) d\ell_s^{(2),\Lambda}(x) \\
&\quad + 2t \sum_{x \in \Lambda} \exp(-2r|x|) + m_t(\Psi),
\end{aligned}$$

where $m_t(\Psi)$ is the martingale with the following expression:

$$m_t(\Psi) = 2 \sum_{i=1,2} \sum_{x \in \Lambda} \exp(-2r|x|) \int_0^t (\phi_s^{(i),\Lambda}(x) - \psi^{(i)}(x)) dw_s^{(i)}(x).$$

Here, since the third term in the right hand side is expressed by

$$\sum_{x \in \Lambda} \exp(-2r|x|) \int_0^t \phi_s^{(1),\Lambda}(x) d\ell_s^{(1),\Lambda}(x) - \sum_{x \in \Lambda} \exp(-2r|x|) \psi^{(1)}(x) \ell_t^{(1),\Lambda}(x)$$

and $\ell_t^{(1),\Lambda}(x)$ is the local time of $\phi_t^{(1),\Lambda}(x)$, this term is nonpositive. Similarly, we also obtain that the fourth term is nonpositive. Using the Schwarz inequality and the Lipschitz continuity of $\Phi = (\phi^{(1)}, \phi^{(2)}) \rightarrow (\Delta \phi^{(1)}, \Delta \phi^{(2)})$ in \mathcal{X}_r^2 , we get

$$(2.18) \quad \|\Phi_t^\Lambda - \Psi\|_r^2 \leq \|\Phi_0^\Lambda - \Psi\|_r^2 + K_1 \int_0^t (1 + \|\Psi_s^\Lambda\|_r^2 + \|\Psi_s^\Lambda - \Psi\|_r^2) ds + m_t(\Psi)$$

with some constant $K_1 > 0$ independent of Λ and $t > 0$. Repeating the argument in the proof of Lemma 2.2 of [4], we obtain (2.16) and (2.17). We can also obtain the bound for Φ_t corresponding to (2.16) and (2.17) by the similar way to the above. \square

3. PROOF OF THE LOWER BOUND

We shall discuss the lower bound in Theorem 1.1. Our goal in this section is the following:

Theorem 3.1. *Under the condition of Theorem 1.1, we have*

$$(3.1) \quad \liminf_{t \rightarrow \infty} E \left[\frac{\phi_t^{(i)}(0)}{\sqrt{\log_d(t)}} \right] \geq \sqrt{C_1/2},$$

$$(3.2) \quad \lim_{t \rightarrow \infty} P(\phi_t^{(i)}(0) \leq \sqrt{(C_i - \epsilon) \log_d(t)}) = 0$$

for every $i = 1, 2$ and $\epsilon > 0$, where constants C_1, C_2 are same as in Theorem 1.1.

We at first prove the lower bound for the expectation (3.1) in Subsection 3.1. As a next step, we establish an estimate for the variance of $\phi_t^{(i)}(0)$ and conclude (3.2) in Subsection 3.2.

3.1. Estimate for the expected value. In this subsection, we establish the lower bound of the expected value (3.1). We at first prepare the following proposition, which plays the key role in the proof of (3.1). It is based on the logarithmic Sobolev inequality for $\mathcal{E}_{\epsilon,\delta}$.

Proposition 3.2. *Let μ_t^N be the law of ϕ_t^N with initial data $\phi_t^N \equiv \phi_t$ on Λ_N and $\phi_t^N \equiv 0$ on $\mathbb{Z}^d \setminus \Lambda_N$. Then, there exist two constants $c_1, c_2 > 0$ such that*

$$d_{\text{var}}(\mu_t^N, \mu_N) \leq c_1 N^d \exp(-c_2 N^{-2}t).$$

Here, d_{var} is the variational distance, that is,

$$d_{\text{var}}(\mu, \nu) := \sup_{A \in \mathcal{B}} |\mu(A) - \nu(A)|,$$

where \mathcal{B} is Borel σ -algebra in $(\mathbb{R}^{\Lambda_N})^2$.

Proof. Verifying the Bakry-Emery criteria (see [5]), we obtain the logarithmic Sobolev inequality for $\mathcal{E}_{\epsilon,\delta}$:

$$H(\mu_t^{N,\epsilon,\delta} | \mu^{N,\epsilon,\delta}) \leq \exp(-cN^{-2}t) H(\mu_0^{N,\epsilon,\delta} | \mu^{N,\epsilon,\delta}),$$

where $\mu_0^{N,\epsilon,\delta}$ is the law of $\phi_t^{N,\epsilon,\delta}$ which is solution of (2.4). Here, we denote the relative entropy by H , that is,

$$H(\mu | \nu) = \begin{cases} E_\nu \left[\frac{d\mu}{d\nu} \log \frac{d\mu}{d\nu} \right], & \text{if } \mu < \nu, \\ \infty, & \text{otherwise.} \end{cases}$$

We therefore get

$$\begin{aligned} d_{\text{var}}(\mu_t^N, \mu_N) &\leq d_{\text{var}}(\mu_t^N, \mu_t^{N,\epsilon}) + d_{\text{var}}(\mu_t^{N,\epsilon}, \mu_t^{N,\epsilon,\delta}) + d_{\text{var}}(\mu_t^{N,\epsilon,\delta}, \mu^{N,\epsilon,\delta}) \\ &\quad + d_{\text{var}}(\mu^{N,\epsilon,\delta}, \mu^{N,\epsilon}) + d_{\text{var}}(\mu^{N,\epsilon}, \mu^N) \\ &\leq \exp(-cN^{-2}t) H(\mu_0^{N,\epsilon,\delta} | \mu^{N,\epsilon,\delta}) + d_{\text{var}}(\mu_t^N, \mu_t^{N,\epsilon}) + d_{\text{var}}(\mu_t^{N,\epsilon}, \mu_t^{N,\epsilon,\delta}) \\ &\quad + d_{\text{var}}(\mu^{N,\epsilon,\delta}, \mu^{N,\epsilon}) + d_{\text{var}}(\mu^{N,\epsilon}, \mu^N). \end{aligned}$$

Here, using (2.6) and (2.5), all terms except the first converge to 0 as $\delta \downarrow 0, \epsilon_1 \downarrow 0$ and $\epsilon_2 \downarrow 0$. Therefore, it suffices to show

$$(3.3) \quad H(\mu_0^{N,\epsilon,\delta} | \mu^{N,\epsilon,\delta}) \leq CN^d$$

with a constant $C > 0$ independent of N, ϵ, δ . Calculating the relative entropy, we obtain

$$\begin{aligned} H(\mu_0^{N,\epsilon,\delta} | \mu^{\epsilon,\delta}) &= 2 \log Z_{\Lambda_N,0} + \log Z_{\Lambda_N,\epsilon,\delta} \\ &\quad + \sum_{x \in \Lambda_N} E_\rho [\log f(\phi^{(1)}(x), \phi^{(2)}(x)) + W_{\epsilon,\delta}(\phi^{(1)}(x), \phi^{(2)}(x))] \\ &\quad + \sum_{i=1,2} E_\rho [H_\Lambda(\phi^{(i)})] \end{aligned}$$

Since $W_{\epsilon,\delta}(u) \equiv 0$ on the support of ρ , we obtain $E_\rho[W_{\epsilon,\delta}(\phi(x))] = 0$ for every $x \in \Lambda_N$. Noting that $W_{\epsilon,\delta}$ is non-negative, we obtain $\log Z_{\Lambda_N,\epsilon,\delta} \leq 0$. Since $N^{-d} \log Z_{\Lambda_N,0}$ converges to $\sigma(0)$ when $N \rightarrow \infty$, where σ is the (unnormalized) surface tension (see [3] or [7]), this term is bounded above by CN^d for some constant $C > 0$. Noting that

$$\sum_{x \in \Lambda_N} E_\rho[\log f(\phi^{(1)}(x), \phi^{(2)}(x))] + \sum_{i=1,2} E_\rho[H_\Lambda(\phi^{(i)})] \leq CN^d$$

holds for some constant $C > 0$, we obtain the desired estimate (3.3). \square

Since we have finished the preparation, we are at the position to show the lower bound for the expected value.

Proof of (3.1). We at first note that

$$\lim_{N \rightarrow \infty} \mu_\Lambda \left(|\Lambda_{\kappa N}|^{-1} \sum_{x \in \Lambda_{\kappa N}} \phi^{(i),N}(x) \leq \sqrt{(C_i - \eta) \log_d(N)} \right) = 0$$

holds for every $i = 1, 2$, $\eta > 0$ and $0 < \kappa < 1$, see [1] (when $d \geq 3$) or [8] (when $d = 2$). Choosing $N = t^{1/2-\epsilon}$ and combining the above with Proposition 3.2, we obtain

$$(3.4) \quad \lim_{t \rightarrow \infty} P \left(|\Lambda_{\kappa N}|^{-1} \sum_{x \in \Lambda_{\kappa N}} \phi_t^{(i),N}(x) \leq \sqrt{(C_i - \eta) \log_d(N)} \right) = 0,$$

for every $i = 1, 2$, $\eta > 0$ and $0 < \kappa < 1$. Since we get

$$\begin{aligned} E \left[\phi_t^{(i)}(0) \right] &= E \left[|\Lambda_{\kappa N}|^{-1} \sum_{x \in \Lambda_{\kappa N}} \phi_t^{(i)}(x) \right] \\ &\geq E \left[|\Lambda_{\kappa N}|^{-1} \sum_{x \in \Lambda_{\kappa N}} \phi_t^{(i),N}(x) \right] \\ &\geq \sqrt{(C_i - \eta) \log_d(N)} P \left(|\Lambda_{\kappa N}|^{-1} \sum_{x \in \Lambda_{\kappa N}} \phi_t^{(i),N}(x) > \sqrt{(C_i - \eta) \log_d(N)} \right). \end{aligned}$$

for every $t > 0$ and $N \geq 1$ from the invariance of the initial data and coefficients of (1.1) under the spatial shift, Proposition 2.1 and the Chebyshev inequality, we obtain

$$(3.5) \quad \liminf_{t \rightarrow \infty} E \left[\frac{\phi_t^{(i)}(0)}{\sqrt{\log_d(t)}} \right] \geq \sqrt{(C_i - \eta)/2}.$$

from (3.4). Since $\eta > 0$ is arbitrary, we obtain the conclusion. \square

3.2. Estimate for the asymptotic variance. Since we have already shown the estimate (3.1) for the expected value, it suffices to show that the variance of $\phi_t^{(1)}(0)$ is small enough in order to obtain (3.2). Our goal in this subsection is the following:

Theorem 3.3. *There exist two constants $K_1, K_2 > 0$ independent of t such that*

$$\text{var}(\phi_t^{(i)}(0)) \leq K_1 \int_0^{K_2 t} p_s(0, 0) ds$$

holds for every $t > 0$ and $i = 1, 2$.

Once we obtain Theorem 3.3, we can immediately conclude (3.2) from (3.1), since

$$\lim_{t \rightarrow \infty} \frac{K_1}{\log_d(t)} \int_0^{K_2 t} p_s(0, 0) ds = 0$$

holds for every $d \geq 2$.

Proof of Theorem 3.3. In [6], the random walk representation for the variance of the dynamics with reflection is established. Once we have the random walk representation for our system similarly to [6], Theorem 3.3 can immediately be obtained as its corollary. However, we shall show Theorem 3.3 using the approximation from the dynamics with weak reflection introduced by (2.3) and (2.4). Applying (2.5), (2.6), (2.7), we only have to show the following: there exist constants $K_1, K_2 > 0$ independent of $\Lambda, \epsilon, \delta$ and t such that

$$(3.6) \quad \text{var}(\phi_t^{(i), \Lambda, \epsilon, \delta}(0)) \leq K_1 \int_0^{K_2 t} p_s(0, 0) ds$$

holds for every $t > 0$ and $i = 1, 2$.

For simple notation, during the proof we denote $\phi_t^{(i), \Lambda, \epsilon, \delta}$ and $\Phi_t^{\Lambda, \epsilon, \delta}$ simply by $\phi_t^{(i)}$ and Φ_t , respectively. Let $F^{(i)} = \phi^{(i)}(0)$. For fixed $t > 0$, we introduce a function $u^{(i)}(s, \Phi)$ by

$$u^{(i)}(s, \Phi) = (P_{t-s} F^{(i)})(\Phi),$$

where P_t is the semigroup associated with the Markov process Φ_t . Then, it is easy to see that $M_s^{(i)} := u^{(i)}(s, \Phi_s)$, $0 \leq s \leq t$ is continuous martingale written as

$$(3.7) \quad M_s^{(i)} = \sqrt{2} \sum_{x \in \Lambda} \sum_{j=1,2} \int_0^s \frac{\partial u^{(i)}}{\partial \phi^{(j)}(x)}(r, \phi_r) dw_r^{(j)}.$$

Note that one can easily see that $u^{(i)}(s, \cdot)$ is a function of C^∞ -class, since $W_{\epsilon, \delta}$ is C^∞ -class. Here, from the definition of M^i , we have

$$\text{var}(\phi_t^{(i)}) = E [\langle M^{(i)} \rangle_t].$$

Using the expression (3.7), we get

$$E [\langle M^{(i)} \rangle_t] = 2 \sum_{x \in \Lambda} \sum_{j=1,2} \int_0^t E \left[\left(\frac{\partial u^{(i)}}{\partial \phi^{(j)}(x)}(r, \phi_r) \right)^2 \right] dr.$$

Here, since Proposition 2.2 implies

$$\sup_{\phi \in (\mathbb{R}^\Lambda)^2} \left| \frac{\partial u^{(i)}}{\partial \phi^{(j)}(x)}(r, \phi) \right| \leq p_{t-r}(0, x),$$

we obtain

$$E [\langle M^{(i)} \rangle_t] \leq 4 \sum_{x \in \Lambda} \int_0^t p_{t-r}(0, x) dr \leq 4 \int_0^t p_{2r}(0, 0) dr,$$

which shows the conclusion. \square

4. PROOF OF THE UPPER BOUND

In this section, we shall establish the upper bound of the height. Our goal is the following:

Theorem 4.1. *Under the condition of Theorem 1.1, we have*

$$(4.1) \quad \limsup_{t \rightarrow \infty} E \left[\frac{\phi_t^{(i)}(0)}{\sqrt{\log_d(t)}} \right] \leq \sqrt{C_i/2},$$

$$(4.2) \quad \lim_{t \rightarrow \infty} P \left(\phi_t^{(1)}(0) \geq \sqrt{(C_i/2 + \epsilon) \log_d(t)} \right) = 0,$$

for every $i = 1, 2$, $\epsilon > 0$, where constants C_1, C_2 are same as in Theorem 1.1.

We shall show Theorem 4.1 by comparing with suitable dynamics. We at first introduce another dynamics of interfaces: the upper interface is on the lower and the lower interface is not pushed down by the upper. Let $\tilde{\Phi}_t = (\tilde{\phi}_t^{(1)}, \tilde{\phi}_t^{(2)})$ be the solution of the following equation

$$(4.3) \quad \begin{cases} d\tilde{\phi}_t^{(1)}(x) = \Delta\tilde{\phi}_t^{(1)}(x)dt + \sqrt{2}dw_t^{(1)}(x) + d\tilde{\ell}_t^{(1)}(x), & x \in \mathbb{Z}^d, \\ d\tilde{\phi}_t^{(2)}(x) = \Delta\tilde{\phi}_t^{(2)}(x)dt + \sqrt{2}dw_t^{(2)}(x) + d\tilde{\ell}_t^{(2)}(x), & x \in \mathbb{Z}^d, \end{cases}$$

where $\tilde{\ell}_t^{(1)}(x)$ and $\tilde{\ell}_t^{(2)}(x)$ are the local times of $\tilde{\phi}_t^{(1)}(x)$ and $\tilde{\phi}_t^{(2)}(x) - \tilde{\phi}_t^{(1)}(x)$ at 0, respectively. The such process can be constructed by the following way: we at first construct the process $\tilde{\phi}_t^{(1)}$, and after that, we construct the process $\tilde{\phi}_t^{(2)}$ over $\tilde{\phi}_t^{(1)}$. The following proposition implies that the system $\tilde{\Phi}_t$ always stays above the original system Φ_t and therefore we only need to establish the upper bound for (4.3).

Proposition 4.2. *Let Φ_t and $\tilde{\Phi}_t$ be solutions of (1.1) and (4.3) with common Brownian motions and initial data, respectively. We then have $\Phi_t \leq \tilde{\Phi}_t$ for every $t > 0$.*

Proof. Let $\tilde{\Phi}_t^\Lambda = (\tilde{\phi}_t^{(1),\Lambda}, \tilde{\phi}_t^{(2),\Lambda})$ be the strong solution of the following SDEs:

$$(4.4) \quad \begin{cases} d\tilde{\phi}_t^{(1)}(x) = \Delta\tilde{\phi}_t^{(1)}(x)dt + \sqrt{2}dw_t^{(1)}(x) + d\tilde{\ell}_t^{(1)}(x), & x \in \Lambda, \\ d\tilde{\phi}_t^{(2)}(x) = \Delta\tilde{\phi}_t^{(2)}(x)dt + \sqrt{2}dw_t^{(2)}(x) + d\tilde{\ell}_t^{(2)}(x), & x \in \Lambda, \\ \tilde{\phi}_t^{(1)}(x) = \tilde{\phi}_t^{(2)}(x) = 0, & x \in \Lambda^c. \end{cases}$$

It is sufficient for the goal to show $\Phi_t^\Lambda \leq \tilde{\Phi}_t^\Lambda$ for every finite $\Lambda \subset \mathbb{Z}^d$, since Φ_t and $\tilde{\Phi}_t$ can be obtained as increasing limits of Φ_t^Λ and $\tilde{\Phi}_t^\Lambda$ as $\Lambda \uparrow \mathbb{Z}^d$, respectively. We define $\psi_t^{(i)}$ by $\psi_t^{(i)} = \phi_t^{(i),\Lambda} - \tilde{\phi}_t^{(i),\Lambda}$. Let us at first show that $\phi_t^{(1),\Lambda} \leq \tilde{\phi}_t^{(1),\Lambda}$. Calculating $(\psi_t^{(1)}(x)^+)^2$, we have

$$\begin{aligned} \left(\psi_t^{(1)}(y)^+ \right)^2 &= 2 \int_0^t \psi_s^{(1)}(y)^+ \Delta \psi_s^{(1)}(y) ds \\ &\quad + 2 \int_0^t \psi_s^{(1)}(y)^+ \left(d\ell_s^{(1),\Lambda}(x) - d\tilde{\ell}_s^{(1),\Lambda}(x) \right) \\ &\quad - 2 \int_0^t \psi_s^{(1)}(y)^+ d\ell_s^{(2),\Lambda}(x) \\ &=: F_1 + F_2 + F_3. \end{aligned}$$

By the definition of $\ell_t^{(1),\Lambda}(x)$, $\ell_s^{(2),\Lambda}(x)$ and $\tilde{\ell}_t^{(1),\Lambda}(x)$, the terms F_2 and F_3 are nonpositive. By the same calculation as in the first step of the proof of Proposition 2.2, we obtain that

$$\sum_{x \in \Lambda} \left(\psi_t^{(1)}(x)^+ \right)^2 = 0,$$

which shows $\phi_t^{(1),\Lambda} \leq \tilde{\phi}_t^{(1),\Lambda}$.

Next, we shall show $\phi_t^{(2),\Lambda} \leq \tilde{\phi}_t^{(2),\Lambda}$. Calculating $(\psi_t^{(1)}(x)^+)^2$, we obtain

$$\begin{aligned} \left(\psi_t^{(2)}(x)^+ \right)^2 &= 2 \int_0^t \psi_s^{(2)}(x)^+ \Delta \psi_s^{(2)}(x) ds \\ &\quad + 2 \int_0^t \psi_s^{(2)}(x)^+ \left(d\ell_s^{(2),\Lambda}(x) - d\tilde{\ell}_s^{(2),\Lambda}(x) \right). \end{aligned}$$

Noting that $\phi_s^{(1),\Lambda} \leq \tilde{\phi}_s^{(1),\Lambda}$, we have

$$\phi_s^{(2),\Lambda}(x) - \phi_s^{(1),\Lambda}(x) > \tilde{\phi}_s^{(2),\Lambda}(x) - \tilde{\phi}_s^{(2),\Lambda}(x)$$

if $\psi_s^{(2)}(x)^+ > 0$ holds. We therefore obtain that the second term is nonpositive. Similarly to above, we conclude $\psi_t^{(2)} = \phi_t^{(2),\Lambda} - \tilde{\phi}_t^{(2),\Lambda} \leq 0$ for every $t > 0$. \square

Theorem 1.1 of [4] indicates the upper bound (4.1) and (4.2) for $i = 1$. We shall focus our attention to the upper bound for $\tilde{\phi}_t^{(2)}$. However, since we also have the lower bound for $\tilde{\phi}_t^{(1)}$ (see [4]), we only need to show the followings bounds for $\tilde{\psi}_t := \tilde{\phi}_t^{(2)} - \tilde{\phi}_t^{(1)}$:

$$(4.5) \quad \limsup_{t \rightarrow \infty} E \left[\frac{\tilde{\psi}_t(0)}{\sqrt{\log_d(t)}} \right] \leq \sqrt{C_2/2} - \sqrt{C_1/2},$$

$$(4.6) \quad \lim_{t \rightarrow \infty} P \left(\tilde{\psi}_t(0) \geq (\sqrt{C_2/2} - \sqrt{C_1/2} + \epsilon) \sqrt{\log_d(t)} \right) = 0,$$

for every $\epsilon > 0$.

We furthermore introduce the stochastic dynamics $\hat{\Phi}_t = (\hat{\phi}_t^{(1)}, \hat{\phi}_t^{(2)})$ by the following SDEs:

$$(4.7) \quad \begin{cases} d\hat{\phi}_t^{(1)}(x) = \Delta \hat{\phi}_t^{(1)}(x) dt + \sqrt{2} dw_t^{(1)}(x), & x \in \mathbb{Z}^d, \\ d\hat{\phi}_t^{(2)}(x) = \Delta \hat{\phi}_t^{(2)}(x) dt + \sqrt{2} dw_t^{(2)}(x) + d\hat{\ell}_t^{(2)}(x), & x \in \mathbb{Z}^d, \end{cases}$$

where $\hat{\ell}_t^{(2)}(x)$ are the local time of $\hat{\phi}_t^{(2)}(x) - \hat{\phi}_t^{(1)}(x)$ at 0. This dynamics can be regarded as the dynamics of interfaces such that one interface is on the other which is dynamics without wall. We shall then compare $\tilde{\psi}_t$ with $\hat{\psi}_t := \hat{\phi}_t^{(2)} - \hat{\phi}_t^{(1)}$. Repeating the argument in the proof of Proposition 4.2, we obtain the following proposition:

Proposition 4.3. *Let $\tilde{\Phi}$ and $\hat{\Phi}$ be solutions of (4.3) and (4.7) with common Brownian motions and initial data, respectively. We then have $\tilde{\psi}_t \leq \hat{\psi}_t$ for every $t > 0$.*

Now, we are at the position to show (4.5) and (4.6). By Proposition 4.3, it is sufficient to show the similar bound for the $\hat{\psi}_t$. We can easily show that $\hat{\psi}_t$ has the same law as $\sqrt{2}\rho_t$, where ρ_t is the solution of the following equation:

$$d\rho_t = \Delta \rho_t(x) dt + \sqrt{2} d\bar{w}_t(x) + d\bar{\ell}_t(x), \quad x \in \mathbb{Z}^d.$$

Here, $\bar{w}_t = \{\bar{w}_t(x); x \in \mathbb{Z}^d\}$ is a family of independent one-dimensional Brownian motions and $\bar{\ell}_t(x)$ is the local time of $\rho_t(x)$ at 0. Applying the upper bound for $\hat{\rho}_t$:

$$\lim_{t \rightarrow \infty} P(\rho_t(0) \geq \sqrt{C' \log_d(t)}) = 0$$

$$\lim_{t \rightarrow \infty} E \left[\frac{\rho_t(0)}{\sqrt{\log_d(t)}} \right] = \sqrt{C_1/2}$$

for every $C' > C_1$ (see [4]), we conclude (4.5) and (4.6) and therefore the upper bound (4.1) and (4.2) for $i = 2$.

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